

THE SCALAR CURVATURE FLOW IN LORENTZIAN MANIFOLDS

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ABSTRACT. We prove the existence of closed hypersurfaces of prescribed scalar curvature in globally hyperbolic Lorentzian manifolds provided there are barriers.

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0. INTRODUCTION

We want to find a closed spacelike hypersurface of prescribed scalar curvature in a globally hyperbolic Lorentzian manifold N having a compact Cauchy hypersurface \mathcal{S}_0 . Looking at the Gauß equation for a spacelike hypersurface M , we deduce that its scalar curvature R satisfies

$$(0.1) \quad R = -[H^2 - |A|^2] + \bar{R} + 2\bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta.$$

Denoting the curvature operator defined by H_2 by F , then this equation is equivalent to

$$(0.2) \quad R = -2F(h_{ij}) + \bar{R} + 2\bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta.$$

Thus, we have to allow that the right-hand side f of the equation

$$(0.3) \quad F|_M = f$$

is defined in $T(N)$, or more precisely, after choosing a local trivialization of $T(N)$, that f depends on $x \in N$ and timelike vectors $\nu \in T_x(N)$, and we look for a closed spacelike hypersurface M satisfying

$$(0.4) \quad F|_M = f(x, \nu) \quad \forall x \in M.$$

In [7] Gerhardts solved this problem by using the method of elliptic regularization. We give a new existence proof, based on the curvature estimates in [10], by showing that the scalar curvature flow exists for all time, and that the leaves $M(t)$ of the flow converge to a solution of (0.4).

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To give a precise statement of the existence result we need a few definitions and assumptions. First, we assume that Ω is a precompact, connected, open subset of N , that is bounded by two compact, connected, spacelike hypersurfaces M_1 and M_2 of class $C^{6,\alpha}$, where M_1 is supposed to lie in the past of M_2 .

Let $F = H_2$ be the scalar curvature operator defined on the open cone $\Gamma_2 \subset \mathbb{R}^n$, and $f = f(x, \nu)$ be of class $C^{4,\alpha}$ in its arguments such that

$$(0.5) \quad 0 < c_1 \leq f(x, \nu) \quad \text{if } \langle \nu, \nu \rangle = -1,$$

$$(0.6) \quad \|f_\beta(x, \nu)\| \leq c_2(1 + \|\nu\|^2),$$

and

$$(0.7) \quad \|f_{\nu^\beta}(x, \nu)\| \leq c_3(1 + \|\nu\|),$$

for all $x \in \bar{\Omega}$ and all past directed timelike vectors $\nu \in T_x(\Omega)$, where $\|\cdot\|$ is a Riemannian reference metric that will be detailed in Section 1.

Remark 0.1. The condition (0.5) is reasonable as is evident from the Einstein equation

$$(0.8) \quad \bar{R}_{\alpha\beta} - \frac{1}{2}\bar{R}\bar{g}_{\alpha\beta} = T_{\alpha\beta},$$

where the energy-momentum tensor $T_{\alpha\beta}$ is supposed to be positive semi-definite for timelike vectors (weak energy condition, cf. [12, p. 89]); but it would be convenient, if the estimate in (0.5) would be valid for all timelike vectors. In fact, we may assume this without loss of generality: Let ϑ be a smooth real function such that

$$(0.9) \quad \frac{c_1}{2} \leq \vartheta \quad \text{and} \quad \vartheta(t) = t \quad \forall t \geq c_1,$$

then, we can replace f by $\vartheta \circ f$ and the new function satisfies our requirements for all timelike vectors. We therefore assume in the following that the relation (0.5) holds for all timelike vectors $\nu \in T_x(N)$ and all $x \in \bar{\Omega}$.

We suppose that the boundary components M_i act as barriers for (F, f) . M_2 is an upper barrier for (F, f) , if M_2 is admissible, i.e. its principal curvatures (κ_i) with respect to the past directed normal belong to Γ_2 , and if

$$(0.10) \quad F|_{M_2} \geq f(x, \nu) \quad \forall x \in M_2.$$

M_1 is a lower barrier for (F, f) , if at the points $\Sigma \subset M_1$, where M_1 is admissible, there holds

$$(0.11) \quad F|_\Sigma \leq f(x, \nu) \quad \forall x \in \Sigma.$$

Σ may be empty.

Now, we can state the main theorem.

Theorem 0.2. *Let M_1 be a lower and M_2 an upper barrier for (F, f) , where $F = H_2$. Then, the problem*

$$(0.12) \quad F|_M = f(x, \nu)$$

has an admissible solution $M \subset \bar{\Omega}$ of class $C^{6,\alpha}$ that can be written as a graph over S_0 provided there exists a strictly convex function $\chi \in C^2(\bar{\Omega})$.

Remark 0.3. As proved in [6, Lemma 2.7] the existence of a strictly convex function χ is guaranteed by the assumption that the level hypersurfaces $\{x^0 = \text{const}\}$ are strictly convex in $\bar{\Omega}$, where (x^α) is a Gaussian coordinate system associated with \mathcal{S}_0 .

Looking at Robertson-Walker space-times it seems that the assumption of the existence of a strictly convex function in the neighbourhood of a given compact set is not too restrictive: in Minkowski space e.g. $\chi = -|x^0|^2 + |x|^2$ is a globally defined strictly convex function.

1. NOTATIONS AND PRELIMINARY RESULTS

The main objective of this section is to state the equations of Gauß, Codazzi, and Weingarten for hypersurfaces M in a $(n+1)$ -dimensional Lorentzian space N . Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta}), (\bar{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by $(g_{ij}), (R_{ijkl})$, etc. Greek indices range from 0 to n and Latin from 1 to n ; the summation convention is always used. Generic coordinate systems in N resp. M will be denoted by (x^α) resp. (ξ^i) . Covariant differentiation will simply be indicated by indices, only in case of possible ambiguity they will be preceded by a semicolon, i.e. for a function u in N , (u_α) will be the gradient and $(u_{\alpha\beta})$ the Hessian, but e.g., the covariant derivative of the curvature tensor will be abbreviated by $\bar{R}_{\alpha\beta\gamma\delta;\epsilon}$. We also point out that

$$(1.1) \quad \bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\epsilon} x_i^\epsilon$$

with obvious generalizations to other quantities.

Let M be a spacelike hypersurface, i.e. the induced metric is Riemannian, with a differentiable normal ν that is timelike. In local coordinates, (x^α) and (ξ^i) , the geometric quantities of the spacelike hypersurface M are connected through the following equations

$$(1.2) \quad x_{ij}^\alpha = h_{ij} \nu^\alpha$$

the so-called Gauß formula. Here, and also in the sequel, a covariant derivative is always a full tensor, i.e.

$$(1.3) \quad x_{ij}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma.$$

The comma indicates ordinary partial derivatives. In this implicit definition the second fundamental form (h_{ij}) is taken with respect to ν . The second equation is the Weingarten equation

$$(1.4) \quad \nu_i^\alpha = h_i^k x_k^\alpha,$$

where we remember that ν_i^α is a full tensor. Finally, we have the Codazzi equation

$$(1.5) \quad h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta} \nu_i^\alpha x_j^\beta x_k^\gamma x_l^\delta$$

and the Gauß equation

$$(1.6) \quad R_{ijkl} = -\{h_{ik}h_{jl} - h_{il}h_{jk}\} + \bar{R}_{\alpha\beta\gamma\delta} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta.$$

Now, let us assume that N is a globally hyperbolic Lorentzian manifold with a compact Cauchy surface. N is then a topological product $\mathbb{R} \times \mathcal{S}_0$, where \mathcal{S}_0 is a compact Riemannian manifold, and there exists a Gaussian coordinate system (x^α) , such that the metric in N has the form

$$(1.7) \quad d\bar{s}_N^2 = e^{2\psi} \{-dx^{02} + \sigma_{ij}(x^0, x) dx^i dx^j\},$$

where σ_{ij} is a Riemannian metric, ψ a function on N , and x an abbreviation for the spacelike components (x^i) , see [1, Lemma 2.2], and [2, Theorem 1.1]. We also

assume that the coordinate system is future oriented, i.e. the time coordinate x^0 increases on future directed curves. Hence, the contravariant timelike vector $(\xi^\alpha) = (1, 0, \dots, 0)$ is future directed as is its covariant version $(\xi_\alpha) = e^{2\psi}(-1, 0, \dots, 0)$.

Let $M = \text{graph } u|_{\mathcal{S}_0}$ be a spacelike hypersurface

$$(1.8) \quad M = \{ (x^0, x) : x^0 = u(x), x \in \mathcal{S}_0 \},$$

then the induced metric has the form

$$(1.9) \quad g_{ij} = e^{2\psi} \{-u_i u_j + \sigma_{ij}\},$$

where σ_{ij} is evaluated at (u, x) , and its inverse $(g^{ij}) = (g_{ij})^{-1}$ can be expressed as

$$(1.10) \quad g^{ij} = e^{-2\psi} \left\{ \sigma^{ij} + \frac{u^i u^j}{v} \right\},$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and

$$(1.11) \quad \begin{aligned} u^i &= \sigma^{ij} u_j \\ v^2 &= 1 - \sigma^{ij} u_i u_j \equiv 1 - |Du|^2. \end{aligned}$$

Hence, $\text{graph } u$ is spacelike if and only if $|Du| < 1$. The covariant form of a normal vector of a graph looks like

$$(1.12) \quad (\nu_\alpha) = \pm v^{-1} e^\psi (1, -u_i).$$

and the contravariant version is

$$(1.13) \quad (\nu^\alpha) = \mp v^{-1} e^{-\psi} (1, u^i).$$

Thus, we have

Remark 1.1. Let M be spacelike graph in a future oriented coordinate system. Then, the contravariant future directed normal vector has the form

$$(1.14) \quad (\nu^\alpha) = v^{-1} e^{-\psi} (1, u^i)$$

and the past directed

$$(1.15) \quad (\nu^\alpha) = -v^{-1} e^{-\psi} (1, u^i).$$

In the Gauß formula (1.2) we are free to choose the future or past directed normal, but we stipulate that we always use the past directed normal. Look at the component $\alpha = 0$ in (1.3) and obtain in view of (1.15)

$$(1.16) \quad e^{-\psi} v^{-1} h_{ij} = -u_{ij} - \bar{\Gamma}_{00}^0 u_i u_j - \bar{\Gamma}_{0j}^0 u_i - \bar{\Gamma}_{0i}^0 u_j - \bar{\Gamma}_{ij}^0.$$

Here, the covariant derivatives are taken with respect to the induced metric of M , and

$$(1.17) \quad -\bar{\Gamma}_{ij}^0 = e^{-\psi} \bar{h}_{ij},$$

where (\bar{h}_{ij}) is the second fundamental form of the hypersurfaces $\{x^0 = \text{const}\}$. An easy calculation shows

$$(1.18) \quad e^{-\psi} \bar{h}_{ij} = -\frac{1}{2} \dot{\sigma}_{ij} - \dot{\psi} \sigma_{ij},$$

where the dot indicates differentiation with respect to x^0 .

Next, let us state under which condition a spacelike hypersurface M can be written as a graph over the Cauchy hypersurface \mathcal{S}_0 . In [15, Lemma 3.1] Kröner proved

Proposition 1.2. *Let N be globally hyperbolic, $\mathcal{S}_0 \subset N$ a compact, connected Cauchy hypersurface, and $M \subset N$ a compact, connected spacelike hypersurface of class C^m , $m \geq 1$. Then, $M = \text{graph } u|_{\mathcal{S}_0}$ with $u \in C^m(\mathcal{S}_0)$.*

Sometimes, we need a Riemannian reference metric, e.g. if we want to estimate tensors. Since the Lorentzian metric can be expressed as

$$(1.19) \quad \bar{g}_{\alpha\beta} dx^\alpha dx^\beta = e^{2\psi} \{-dx^0{}^2 + \sigma_{ij} dx^i dx^j\},$$

we define a Riemannian reference metric $(\tilde{g}_{\alpha\beta})$ by

$$(1.20) \quad \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta = e^{2\psi} \{dx^0{}^2 + \sigma_{ij} dx^i dx^j\}$$

and we abbreviate the corresponding norm of a vectorfield η by

$$(1.21) \quad \|\eta\| = (\tilde{g}_{\alpha\beta} \eta^\alpha \eta^\beta)^{1/2},$$

with similar notations for higher order tensors.

For a spacelike hypersurface $M = \text{graph } u$ the induced metrics with respect to $(\bar{g}_{\alpha\beta})$ resp. $(\tilde{g}_{\alpha\beta})$ are related as follows

$$(1.22) \quad \begin{aligned} \tilde{g}_{ij} &= \tilde{g}_{\alpha\beta} x_i^\alpha x_j^\beta = e^{2\psi} [u_i u_j + \sigma_{ij}] \\ &= g_{ij} + 2e^{2\psi} u_i u_j. \end{aligned}$$

Thus, if $(\xi^i) \in T_p(M)$ is a unit vector for (g_{ij}) , then

$$(1.23) \quad \tilde{g}_{ij} \xi^i \xi^j = 1 + 2e^{2\psi} |u_i \xi^i|^2,$$

and we conclude for future reference

Lemma 1.3. *Let $M = \text{graph } u$ be a spacelike hypersurface in N , $p \in M$, and $\xi \in T_p(M)$ a unit vector, then*

$$(1.24) \quad \|x_i^\beta \xi^i\| \leq c(1 + |u_i \xi^i|) \leq c\tilde{v},$$

where $\tilde{v} = v^{-1}$.

2. CURVATURE FUNCTIONS

Let $\Gamma \subset \mathbb{R}^n$ be an open cone containing the positive cone Γ_+ , and $F \in C^{2,\alpha}(\Gamma) \cap C^0(\bar{\Gamma})$ a positive symmetric function satisfying the condition

$$(2.1) \quad F_i = \frac{\partial F}{\partial \kappa^i} > 0,$$

then, F can also be viewed as a function defined on the space of symmetric matrices \mathcal{S}_Γ , the eigenvalues of which belong to Γ , namely, let $(h_{ij}) \in \mathcal{S}_\Gamma$ with eigenvalues κ_i , $1 \leq i \leq n$, then define F on \mathcal{S}_Γ by

$$(2.2) \quad F(h_{ij}) = F(\kappa_i).$$

If we define

$$(2.3) \quad F^{ij} = \frac{\partial F}{\partial h_{ij}}$$

and

$$(2.4) \quad F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}}$$

then,

$$(2.5) \quad F^{ij} \xi_i \xi_j = \frac{\partial F}{\partial \kappa_i} |\xi^i|^2 \quad \forall \xi \in \mathbb{R}^n,$$

in an appropriate coordinate system,

$$(2.6) \quad F^{ij} \text{ is diagonal if } h_{ij} \text{ is diagonal,}$$

and

$$(2.7) \quad F^{ij,kl} \eta_{ij} \eta_{kl} = \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j} \eta_{ii} \eta_{jj} + \sum_{i \neq j} \frac{F_i - F_j}{\kappa_i - \kappa_j} (\eta_{ij})^2,$$

for any $(\eta_{ij}) \in \mathcal{S}$, where \mathcal{S} is the space of all symmetric matrices. The second term on the right-hand side of (2.7) is non-positive if F is concave, and non-negative if F is convex, and has to be interpreted as a limit if $\kappa_i = \kappa_j$.

The preceding considerations are also applicable if the κ_i are the principal curvatures of a spacelike hypersurface M with metric (g_{ij}) . F can then be looked at as being defined on the space of all symmetric tensors (h_{ij}) the eigenvalues of which belong to Γ . Such tensors will be called admissible; when the second fundamental form of M is admissible, then, we also call M admissible.

For an admissible tensor (h_{ij})

$$(2.8) \quad F^{ij} = \frac{\partial F}{\partial h_{ij}}$$

is a contravariant tensor of second order. Sometimes it will be convenient to circumvent the dependence on the metric by considering F to depend on the mixed tensor

$$(2.9) \quad h_j^i = g^{ik} h_{kj}.$$

Then,

$$(2.10) \quad F_i^j = \frac{\partial F}{\partial h_j^i}$$

is also a mixed tensor with contravariant index j and covariant index i . Such functions F are called curvature functions.

Important examples are the elementary symmetric polynomials of order k , H_k , $1 \leq k \leq n$,

$$(2.11) \quad H_k(\kappa_i) = \sum_{i_1 < \dots < i_k} \kappa_{i_1} \dots \kappa_{i_k}.$$

They are defined on an open set Γ_k that can be characterized as the connected component of $\{H_k > 0\}$ that contains the positive cone Γ_+ . The Γ_k are cones, $\Gamma_n = \Gamma_+$, and in [4] it is proved that

$$(2.12) \quad \Gamma_k \subset \Gamma_{k-1}.$$

Huisken and Sinestrari in [13, Section 2] gave an equivalent characterisation of Γ_k by showing that

$$(2.13) \quad \Gamma_k = \{(\kappa_i) \in \mathbb{R}^n : H_1(\kappa_i) > 0, H_2(\kappa_i) > 0, \dots, H_k(\kappa_i) > 0\}.$$

They also proved that Γ_k is convex. The H_k are strictly monotone in Γ_k , cf. [13, Lemma 2.4], and the k -th roots

$$(2.14) \quad \sigma_k = H_k^{\frac{1}{k}}$$

are also concave, cf. [16].

Since we have in mind that the κ_i are the principal curvatures of a hypersurface, we use the standard symbols H and $|A|$ for

$$(2.15) \quad H = \sum_i \kappa_i,$$

and

$$(2.16) \quad |A|^2 = \sum_i \kappa_i^2.$$

We note that

$$(2.17) \quad \frac{1}{n} H^2 \leq |A|^2.$$

The scalar curvature function $F = H_2$ can be expressed as

$$(2.18) \quad F = \frac{1}{2}(H^2 - |A|^2),$$

and we deduce that for $(\kappa_i) \in \Gamma_2$

$$(2.19) \quad |A|^2 \leq H^2,$$

$$(2.20) \quad F \leq \frac{1}{2} H^2,$$

$$(2.21) \quad F_i = H - \kappa_i,$$

and hence,

$$(2.22) \quad H > \kappa_i,$$

$$(2.23) \quad H F_i \geq F,$$

for (2.23) is equivalent to

$$(2.24) \quad H \kappa_i \leq \frac{1}{2} H^2 + \frac{1}{2} |A|^2,$$

which is obviously valid.

3. THE EVOLUTION PROBLEM

To prove the existence of hypersurfaces of prescribed curvature F for $F = \sigma_2$ we look at the evolution problem

$$(3.1) \quad \begin{aligned} \dot{x} &= (F - f)\nu, \\ x(0) &= x_0, \end{aligned}$$

where ν is the past-directed normal of the flow hypersurfaces $M(t)$, $F = \sigma_2$ the curvature evaluated at $M(t)$, $x = x(t)$ an embedding and x_0 an embedding of an initial hypersurface M_0 , which we choose to be the upper barrier M_2 .

Since F is an elliptic operator, short-time existence, and hence, existence in a maximal time interval $[0, T^*)$ is guaranteed, cf. [9]. If we are able to prove uniform a priori estimates in $C^{2,\alpha}$, long-time existence and convergence to a stationary solution will follow immediately.

But before we prove the a priori estimates, we want to show how the metric, the second fundamental form, and the normal vector of the hypersurfaces $M(t)$ evolve. All time derivatives are total derivatives. We shall omit the proofs, which can be found in [9, Chapter 2.3].

Lemma 3.1. *The metric, the normal vector and the second fundamental form of $M(t)$ satisfy the evolution equations*

$$(3.2) \quad \dot{g}_{ij} = 2(F - f)h_{ij},$$

$$(3.3) \quad \dot{\nu} = \nabla_M(F - f) = g^{ij}(F - f)_i x_j,$$

$$(3.4) \quad \dot{h}_i^j = (F - f)_i^j - (F - f)h_i^k h_k^j - (F - f)\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_k^\delta g^{kj},$$

$$(3.5) \quad \dot{h}_{ij} = (F - f)_{ij} + (F - f)h_i^k h_{kj} - (F - f)\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta.$$

Lemma 3.2. *The term $(F - f)$ evolves according to the equation*

$$(3.6) \quad (F - f)' - F^{ij}(F - f)_{ij} = -F^{ij}h_{ik}h_j^k(F - f) - f_\alpha\nu^\alpha(F - f) \\ - f_{\nu^\alpha}x_i^\alpha(F - f)_j g^{ij} - F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta(F - f).$$

From (3.4) we deduce with the help of the Ricci identities a parabolic equation for the second fundamental form, cf. [9, Lemma 2.4.9].

Lemma 3.3. *The mixed tensor h_i^j satisfies the parabolic equation*

$$(3.7) \quad \begin{aligned} & \dot{h}_i^j - F^{kl}h_{i;kl}^j \\ &= -F^{kl}h_{rk}h_l^r h_i^j + f h_i^k h_k^j \\ & \quad - f_{\alpha\beta}x_i^\alpha x_k^\beta g^{kj} - f_\alpha\nu^\alpha h_i^j - f_{\alpha\nu^\beta}(x_i^\alpha x_k^\beta h^{kj} + x_l^\alpha x_k^\beta h_i^k g^{lj}) \\ & \quad - f_{\nu^\alpha\nu^\beta}x_l^\alpha x_k^\beta h_i^k h^{lj} - f_{\nu^\beta}x_k^\beta h_{i;l}^k g^{lj} - f_{\nu^\alpha}\nu^\alpha h_i^k h_k^j \\ & \quad + F^{kl,rs}h_{kl;i}h_{rs}^j + 2F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x_m^\alpha x_i^\beta x_r^\gamma x_l^\delta h_m^r g^{rj} \\ & \quad - F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x_m^\alpha x_k^\beta x_r^\gamma x_l^\delta h_i^m g^{rj} - F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x_m^\alpha x_k^\beta x_i^\gamma x_l^\delta h^m j \\ & \quad - F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_k^\beta \nu^\gamma x_l^\delta h_i^j + f\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_m^\delta g^{mj} \\ & \quad + F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\{\nu^\alpha x_k^\beta x_l^\gamma x_i^\delta x_m^\epsilon g^{mj} + \nu^\alpha x_i^\beta x_k^\gamma x_m^\delta x_l^\epsilon g^{mj}\}. \end{aligned}$$

Remark 3.4. In view of the maximum principle, we immediately deduce from (3.6) that the term $(F - f)$ has a sign during the evolution if it has one at the beginning, i.e., if the starting hypersurface M_0 is the upper barrier M_2 , then $(F - f)$ is non-negative

$$(3.8) \quad F \geq f.$$

4. LOWER ORDER ESTIMATES

Since the two boundary components M_1, M_2 of $\partial\Omega$ are compact, connected spacelike hypersurfaces, they can be written as graphs over the Cauchy hypersurface \mathcal{S}_0 , $M_i = \text{graph } u_i$, $i = 1, 2$, and we have

$$(4.1) \quad u_1 \leq u_2,$$

for M_1 should lie in the past of M_2 .

Let us look at the evolution equation (3.1) with initial hypersurface M_0 equal to M_2 defined on a maximal time interval $I = [0, T^*)$, $T^* \leq \infty$. Since the initial hypersurface is a graph over \mathcal{S}_0 , we can write

$$(4.2) \quad M(t) = \text{graph } u(t)|_{\mathcal{S}_0} \quad \forall t \in I,$$

where u is defined in the cylinder $Q_{T^*} = I \times \mathcal{S}_0$.

We then deduce from (3.1), looking at the component $\alpha = 0$, that u satisfies a parabolic equation of the form

$$(4.3) \quad \dot{u} = -e^{-\psi} v^{-1} (F - f),$$

where we use the notations in Section 2, and where we emphasize that the time derivative is a total derivative, i.e.

$$(4.4) \quad \dot{u} = \frac{\partial u}{\partial t} + u_i \dot{x}^i.$$

Since the past directed normal can be expressed as

$$(4.5) \quad (\nu^\alpha) = -e^{-\psi} v^{-1} (1, u^i),$$

we conclude from (3.1), (4.3) and (4.4)

$$(4.6) \quad \frac{\partial u}{\partial t} = -e^{-\psi} v (F - f).$$

Thus, $\frac{\partial u}{\partial t}$ is non-positive in view of Remark 3.4.

Next, let us state our first a priori estimate, [9, Theorem 2.7.9].

Lemma 4.1. *Suppose that the boundary components act as barriers for (F, f) , then the flow hypersurfaces stay in $\bar{\Omega}$ during the evolution.*

For the C^1 -estimate the term $\tilde{v} = v^{-1}$ is of great importance. It satisfies the following evolution equation.

Lemma 4.2. *Consider the flow (3.1) in the distinguished coordinate system associated with \mathcal{S}_0 . Then, \tilde{v} satisfies the evolution equation*

$$(4.7) \quad \begin{aligned} \dot{\tilde{v}} - F^{ij} \tilde{v}_{ij} &= -F^{ij} h_{ik} h_j^k \tilde{v} - f \eta_{\alpha\beta} \nu^\alpha \nu^\beta \\ &\quad - 2F^{ij} h_j^k x_i^\alpha x_k^\beta \eta_{\alpha\beta} - F^{ij} \eta_{\alpha\beta\gamma} x_i^\beta x_j^\gamma \nu^\alpha \\ &\quad - F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_k^\gamma x_j^\delta \eta_\epsilon x_l^\epsilon g^{kl} \\ &\quad - f_\beta x_i^\beta x_k^\alpha \eta_{\alpha\gamma} g^{ik} - f_{\nu\beta} x_k^\beta h^{ik} x_i^\alpha \eta_\alpha, \end{aligned}$$

where η is the covariant vector field $(\eta_\alpha) = e^\psi(-1, 0, \dots, 0)$.

The proof uses the relation

$$(4.8) \quad \tilde{v} = \eta_\alpha \nu^\alpha$$

and is identical to that of [6, Lemma 4.4] having in mind that presently f also depends on ν .

Lemma 4.3. *Let $M(t) = \text{graph } u(t)$ be the flow hypersurfaces, then, we have*

$$(4.9) \quad \dot{u} - F^{ij} u_{ij} = e^{-\psi} \tilde{v} f + \bar{F}_{00}^0 F^{ij} u_i u_j + 2F^{ij} \bar{F}_{0i}^0 u_j + F^{ij} \bar{F}_{ij}^0,$$

where all covariant derivatives are taken with respect to the induced metric of the flow hypersurfaces, and the time derivative \dot{u} is the total time derivative, i.e. it is given by (4.4).

Proof. We use the relation (4.3) together with (1.16). □

As an immediate consequence we obtain

Lemma 4.4. *The composite function*

$$(4.10) \quad \varphi = e^{\mu e^{\lambda u}},$$

where μ, λ are constants, satisfies the equation

$$(4.11) \quad \begin{aligned} \dot{\varphi} - F^{ij} \varphi_{ij} = & f e^{-\psi} \tilde{v} \mu \lambda e^{\lambda u} \varphi + F^{ij} u_i u_j \bar{F}_{00}^0 \mu \lambda e^{\lambda u} \varphi \\ & + 2 F^{ij} u_i \bar{F}_{0j}^0 \mu \lambda e^{\lambda u} \varphi + F^{ij} \bar{F}_{ij}^0 \mu \lambda e^{\lambda u} \varphi \\ & - [1 + \mu e^{\lambda u}] F^{ij} u_i u_j \mu \lambda^2 e^{\lambda u} \varphi. \end{aligned}$$

Before we can prove the C^1 - estimates we need two more lemmata.

Lemma 4.5. *There is a constant $c = c(\Omega)$ such that for any positive function $0 < \epsilon = \epsilon(x)$ on \mathcal{S}_0 and any hypersurface $M(t)$ of the flow we have*

$$(4.12) \quad \|\nu\| \leq c \tilde{v},$$

$$(4.13) \quad g^{ij} \leq c \tilde{v}^2 \sigma^{ij},$$

$$(4.14) \quad F^{ij} \leq F^{kl} g_{kl} g^{ij},$$

$$(4.15) \quad |F^{ij} h_j^k x_i^\alpha x_k^\beta \eta_{\alpha\beta}| \leq \frac{\epsilon}{2} F^{ij} h_i^k h_{kj} \tilde{v} + \frac{c}{2\epsilon} F^{ij} g_{ij} \tilde{v}^3,$$

$$(4.16) \quad |F^{ij} \eta_{\alpha\beta\gamma} x_i^\beta x_j^\gamma \nu^\alpha| \leq c \tilde{v}^3 F^{ij} g_{ij},$$

$$(4.17) \quad |F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_k^\gamma x_j^\delta \eta_\epsilon x_l^\epsilon g^{kl}| \leq c \tilde{v}^3 F^{ij} g_{ij}.$$

Lemma 4.6. *Let $M \subset \bar{\Omega}$ be a graph over \mathcal{S}_0 , $M = \text{graph } u$, and $\epsilon = \epsilon(x)$ a function defined on \mathcal{S}_0 , $0 < \epsilon < \frac{1}{2}$. Let φ be defined through*

$$(4.18) \quad \varphi = e^{\mu e^{\lambda u}},$$

where $0 < \mu$ and $\lambda < 0$. Then, there exists $c = c(\Omega)$ such that

$$(4.19) \quad \begin{aligned} 2|F^{ij} \tilde{v}_i \varphi_j| \leq & c F^{ij} g_{ij} \tilde{v}^3 |\lambda| \mu e^{\lambda u} \varphi + (1 - 2\epsilon) F^{ij} h_i^k h_{kj} \tilde{v} \varphi \\ & + \frac{1}{1 - 2\epsilon} F^{ij} u_i u_j \mu^2 \lambda^2 e^{2\lambda u} \tilde{v} \varphi. \end{aligned}$$

A proof of Lemma 4.5 and Lemma 4.6 can be found in [7].

Applying Lemma 4.5 to the evolution equation for \tilde{v} we conclude

Lemma 4.7. *There exists a constant $c = c(\Omega)$ such that for any function ϵ , $0 < \epsilon = \epsilon(x) < 1$, defined on \mathcal{S}_0 the term \tilde{v} satisfies an evolution inequality of the form*

$$(4.20) \quad \begin{aligned} \dot{\tilde{v}} - F^{ij} \tilde{v}_{ij} \leq & -(1 - \epsilon) F^{ij} h_i^k h_{kj} \tilde{v} - f \eta_{\alpha\beta} \nu^\alpha \nu^\beta \\ & + \frac{c}{\epsilon} F^{ij} g_{ij} \tilde{v}^3 + c \|f_\beta\| \tilde{v}^2 + f_{\nu^\beta} x_l^\beta h^{kl} u_k e^\psi. \end{aligned}$$

We are now ready to prove the uniform boundedness of \tilde{v} .

Proposition 4.8. *Assume that there are positive constants c_i , $1 \leq i \leq 3$, such that for any $x \in \Omega$ and any past directed timelike vector ν there holds*

$$(4.21) \quad -c_1 \leq f(x, \nu),$$

$$(4.22) \quad \|f_\beta(x, \nu)\| \leq c_2(1 + \|\nu\|),$$

and

$$(4.23) \quad \|f_{\nu\beta}(x, \nu)\| \leq c_3.$$

Then, the term \tilde{v} remains uniformly bounded during the evolution

$$(4.24) \quad \tilde{v} \leq c = c(\Omega, c_1, c_2, c_3).$$

Proof. We show that the function

$$(4.25) \quad w = \tilde{v}\varphi,$$

φ as in (4.18), is uniformly bounded, if we choose

$$(4.26) \quad 0 < \mu < 1 \quad \text{and} \quad \lambda < -1,$$

appropriately, and assume furthermore, without loss of generality, that $u \leq -1$, for otherwise replace u by $(u - c)$, c large, in the definition of φ . With the help of Lemma 4.4, Lemma 4.6 and Lemma 4.7 we derive from the relation

$$(4.27) \quad \dot{w} - F^{ij}w_{ij} = [\dot{\tilde{v}} - F^{ij}\tilde{v}_{ij}]\varphi + [\dot{\varphi} - F^{ij}\varphi_{ij}]\tilde{v} - 2F^{ij}\tilde{v}_i\varphi_j$$

the parabolic inequality

$$(4.28) \quad \begin{aligned} \dot{w} - F^{ij}w_{ij} &\leq -\epsilon F^{ij}h_i^k h_{kj}\tilde{v}\varphi + c[\epsilon^{-1} + |\lambda|\mu e^{\lambda u}]F^{ij}g_{ij}\tilde{v}^3\varphi \\ &\quad + [\frac{1}{1-2\epsilon} - 1]F^{ij}u_i u_j \mu^2 \lambda^2 e^{2\lambda u} \tilde{v}\varphi \\ &\quad - F^{ij}u_i u_j \mu \lambda^2 e^{\lambda u} \tilde{v}\varphi \\ &\quad + f[-\eta_{\alpha\beta}\nu^\alpha \nu^\beta + e^{-\psi}\mu\lambda e^{\lambda u}\tilde{v}^2]\varphi \\ &\quad + c\|f_\beta\|\tilde{v}^2\varphi + f_{\nu\beta}x_l^\beta h^{kl}u_k e^\psi\varphi, \end{aligned}$$

where we have chosen the same function $\epsilon = \epsilon(x)$ in Lemma 4.6 resp. Lemma 4.7. We claim that w is uniformly bounded provided μ and λ are chosen appropriately. We shall use the maximum principle, therefore let $0 < T < T^*$ and $x_0 = x(t_0, \xi_0)$ be such that

$$(4.29) \quad \sup_{[0, T]} \sup_{M(t)} w = w(t_0, \xi_0).$$

To exploit the good term

$$(4.30) \quad -\epsilon F^{ij}h_i^k h_{kj}\tilde{v}\varphi,$$

we use the fact that $Dw(x_0) = 0$, or, equivalently

$$(4.31) \quad \begin{aligned} -\tilde{v}_i &= \mu\lambda e^{\lambda u}\tilde{v}u_i \\ &= e^\psi h_i^k u_k - \eta_{\alpha\beta}\nu^\alpha x_i^\beta, \end{aligned}$$

where the second equation follows from (4.8) and the definition of the covariant vectorfield $\eta = e^\psi(-1, 0, \dots, 0)$. Next, we choose a coordinate system (ξ^i) such that in the critical point

$$(4.32) \quad g_{ij} = \delta_{ij} \quad \text{and} \quad h_i^k = \kappa_i \delta_i^k,$$

and the labelling of the principal curvatures corresponds to

$$(4.33) \quad \kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n.$$

Then, we deduce from (4.31)

$$(4.34) \quad e^\psi \kappa_i u_i = \mu \lambda e^{\lambda u} \tilde{v} u_i + \eta_{\alpha\beta} \nu^\alpha x_i^\beta.$$

Assume that $\tilde{v}(x_0) \geq 2$, and let $i = i_0$ be an index such that

$$(4.35) \quad |u_{i_0}|^2 \geq \frac{1}{n} \|Du\|^2.$$

Setting $(e^i) = \frac{\partial}{\partial \xi^{i_0}}$ and assuming without loss of generality that $0 < u_i e^i$ in x_0 we infer from Lemma 1.3

$$(4.36) \quad \begin{aligned} e^\psi \kappa_{i_0} u_i e^i &= \mu \lambda e^{\lambda u} \tilde{v} u_i e^i + \eta_{\alpha\beta} \nu^\alpha x_i^\beta e^i \\ &\leq \mu \lambda e^{\lambda u} \tilde{v} u_i e^i + c \tilde{v}^2, \end{aligned}$$

and we deduce further in view of (1.10), (1.11) and (4.35) that

$$(4.37) \quad \kappa_{i_0} \leq [\mu \lambda e^{\lambda u} + c] \tilde{v} e^{-\psi} \leq \frac{1}{2} \mu \lambda e^{\lambda u} \tilde{v} e^{-\psi},$$

if $|\lambda|$ is sufficiently large, i.e. κ_{i_0} is negative and of the same order as \tilde{v} . The Weingarten equation and Lemma 1.3 yield

$$(4.38) \quad \|\nu_i^\beta u^i\| = \|h_i^k u^i x_k^\beta\| \leq c \tilde{v} [h_i^k u^i h_{kl} u^l]^{\frac{1}{2}},$$

and therefore, we infer from (4.31)

$$(4.39) \quad \|\nu_i^\beta u^i\| \leq c \mu |\lambda| e^{\lambda u} \tilde{v}^3$$

in critical points of w , and hence, that in those points, the term involving $f_{\tilde{v}\beta}$ on the right-hand side of inequality (4.28) can be estimated from above by

$$(4.40) \quad |f_{\nu^\beta} \nu_i^\beta u^i e^\psi \varphi| \leq c c_3 \mu |\lambda| e^{\lambda u} \tilde{v}^3 \varphi.$$

Next, let us estimate the crucial term in (4.30). Using the particular coordinate system (4.32), as well as the inequalities (4.33), together with the fact that κ_{i_0} is negative, we conclude

$$(4.41) \quad -F^{ij} h_i^k h_{kj} \leq -\sum_{i=1}^{i_0} F_i^i \kappa_i^2 \leq -\sum_{i=1}^{i_0} F_i^i \kappa_{i_0}^2.$$

F is concave, and therefore, we have in view of (4.33)

$$(4.42) \quad F_1^1 \geq F_2^2 \geq \cdots \geq F_n^n,$$

cf. [3, Lemma 2]. Hence, we conclude

$$(4.43) \quad -\sum_{i=1}^{i_0} F_i^i \leq -F_1^1 \leq -\frac{1}{n} \sum_{i=1}^n F_i^i.$$

Using (4.37), (4.41), (4.43), (2.20) and (2.21), we deduce further

$$(4.44) \quad \begin{aligned} -F^{ij} h_i^k h_{kj} &\leq -c F^{ij} g_{ij} \mu^2 \lambda^2 e^{2\lambda u} \tilde{v}^2 \\ &\leq -c \mu^2 \lambda^2 e^{2\lambda u} \tilde{v}^2 \end{aligned}$$

Inserting this estimate, and the estimate in (4.40) in (4.28), with $\epsilon = e^{-\lambda u}$, we obtain

$$\begin{aligned}
 \dot{w} - F^{ij} w_{ij} &\leq -cF^{ij} g_{ij} \mu^2 \lambda^2 e^{\lambda u} \tilde{v}^3 \varphi + cF^{ij} g_{ij} \mu |\lambda| e^{\lambda u} \tilde{v}^3 \varphi \\
 (4.45) \quad &+ \frac{2}{1-2\epsilon} F^{ij} u_i u_j \mu^2 \lambda^2 e^{\lambda u} \tilde{v} \varphi - F^{ij} u_i u_j \mu \lambda^2 e^{\lambda u} \tilde{v} \varphi \\
 &+ cc_1 \mu |\lambda| e^{\lambda u} \tilde{v}^2 \varphi + cc_2 \tilde{v}^3 \varphi + cc_3 \mu |\lambda| e^{\lambda u} \tilde{v}^3 \varphi,
 \end{aligned}$$

where $|\lambda|$ is chosen so large that

$$(4.46) \quad e^{-\lambda u} \leq \frac{1}{4}.$$

Choosing $\mu = \frac{1}{4}$ and $|\lambda|$ sufficient large, we see that in view of (4.44) the right-hand side of the preceding inequality is negative, contradicting the maximum principle, i.e. the maximum of w cannot occur at a point where $\tilde{v} \geq 2$. Thus, the desired uniform estimate for w and hence \tilde{v} is proved. \square

Remark 4.9. Notice that the preceding C^1 -estimate is valid for any curvature function F that is monotone, concave and homogeneous of degree 1.

Let us close this section with an interesting observation that is an immediate consequence of the preceding proof, we have especially (4.41) and (4.43) in mind.

Lemma 4.10. *Suppose $F = \sigma_2$ is evaluated at a point (κ_i) , and assume that κ_{i_0} is a component that is either negative or the smallest component of that particular n -tupel, then*

$$(4.47) \quad \sum_{i=1}^n F_i \kappa_i^2 \geq \frac{1}{n} \sum_{i=1}^n F_i \kappa_{i_0}^2.$$

5. CURVATURE ESTIMATES

We want to prove that the principal curvatures of the flow hypersurfaces are uniformly bounded. Let us first prove an a priori estimate for F .

Lemma 5.1. *Let $M(t)$, $0 \leq t < T^*$, be solutions of the evolution problem (3.1) with $M(0) = M_2$ and $F = \sigma_2$. Then, F is bounded from above during the evolution provided the $M(t)$ are uniformly spacelike, i.e. uniform C^1 -estimates are valid.*

Proof. Let $0 < T < T^*$ and $x_0 = x(t_0, \xi_0)$ be such that

$$(5.1) \quad \sup_{[0, T]} \sup_{M(t)} (F - f) = (F - f)(x_0) > 0.$$

Applying the maximum principle we deduce from (3.6)

$$(5.2) \quad 0 \leq -F^{ij} h_{ik} h^{kj} + c(1 + F^{ij} g_{ij}),$$

where we have estimated bounded terms by a constant c .

Then, we infer from (2.17), (2.21) and (2.23)

$$(5.3) \quad 0 \leq -\frac{1}{2n} FH + c(1 + F^{-1}H),$$

which is equivalent to

$$(5.4) \quad 0 \leq -\frac{1}{2n}F^2 + c(FH^{-1} + 1).$$

Thus, in view of (2.20), we obtain an a priori estimate for F . \square

Remark 5.2. Let χ be the strictly convex function. Its evolution equation is

$$(5.5) \quad \begin{aligned} \dot{\chi} - F^{ij}\chi_{ij} &= f\chi_\alpha\nu^\alpha - F^{ij}\chi_{\alpha\beta}x_i^\alpha x_j^\beta \\ &\leq f\chi_\alpha\nu^\alpha - c_0F^{ij}g_{ij}, \end{aligned}$$

where $c_0 > 0$ is independent of t .

Proposition 5.3. *Under the assumptions of Lemma 5.1 the principal curvatures $\kappa_i, 1 \leq i \leq n$, of the flow hypersurfaces are uniformly bounded during the evolution provided there exists a strictly convex function $\chi \in C^2(\bar{\Omega})$.*

Proof. Let ζ and w be respectively defined by

$$(5.6) \quad \zeta = \sup\{h_{ij}\eta^i\eta^j : \|\eta\| = 1\},$$

$$(5.7) \quad w = \log \zeta + \lambda\chi,$$

where $\lambda > 0$ is supposed to be large. We claim that w is bounded, if λ is chosen sufficiently large.

Let $0 < T < T^*$, and $x_0 = x_0(t_0)$, with $0 < t_0 \leq T$, be a point in $M(t_0)$ such that

$$(5.8) \quad \sup_{M_0} w < \sup\{\sup_{M(t)} w : 0 < t \leq T\} = w(x_0).$$

We then introduce a Riemannian normal coordinate system (ξ^i) at $x_0 \in M(t_0)$ such that at $x_0 = x(t_0, \xi_0)$ we have

$$(5.9) \quad g_{ij} = \delta_{ij} \quad \text{and} \quad \zeta = h_n^n.$$

Let $\tilde{\eta} = (\tilde{\eta}^i)$ be the contravariant vector field defined by

$$(5.10) \quad \tilde{\eta} = (0, \dots, 0, 1),$$

and set

$$(5.11) \quad \tilde{\zeta} = \frac{h_{ij}\tilde{\eta}^i\tilde{\eta}^j}{g_{ij}\tilde{\eta}^i\tilde{\eta}^j}.$$

$\tilde{\zeta}$ is well defined in neighbourhood of (t_0, ξ_0) .

Now, define \tilde{w} by replacing ζ by $\tilde{\zeta}$ in (5.7); then, \tilde{w} assumes its maximum at (t_0, ξ_0) .

Moreover, at (t_0, ξ_0) we have

$$(5.12) \quad \dot{\tilde{\zeta}} = \dot{h}_n^n,$$

and the spatial derivatives do also coincide; in short, at (t_0, ξ_0) $\tilde{\zeta}$ satisfies the same differential equation (3.7) as h_n^n . For the sake of greater clarity, let us therefore treat h_n^n like a scalar and pretend that w is defined by

$$(5.13) \quad w = \log h_n^n + \lambda\chi.$$

We assume that the section curvatures are labelled according to (4.33).

At (t_0, ξ_0) we have $\dot{w} \geq 0$, and, in view of the maximum principle, we deduce from (3.7), (5.5), (2.7) and (4.42)

$$(5.14) \quad \begin{aligned} 0 \leq & -\frac{1}{2}F^{ij}h_{ki}h_j^k + ch_n^n + cF^{ij}g_{ij} + \lambda c - \lambda c_0 F^{ij}g_{ij} \\ & + F^{ij}(\log h_n^n)_i(\log h_n^n)_j + \frac{2}{\kappa_n - \kappa_1} \sum_{i=1}^n (F_n - F_i)(h_{ni};^n)^2 (h_n^n)^{-1}, \end{aligned}$$

where we have estimated bounded terms by a constant c , and assumed that h_n^n and λ are larger than 1. We distinguish two cases

Case 1. Suppose that

$$(5.15) \quad |\kappa_1| \geq \epsilon_1 \kappa_n,$$

where $\epsilon_1 > 0$ is small. Then, we infer from Lemma 4.10

$$(5.16) \quad F^{ij}h_{ki}h_j^k \geq \frac{1}{n}F^{ij}g_{ij}\epsilon_1^2\kappa_n^2,$$

and

$$(5.17) \quad F^{ij}g_{ij} \geq F(1, \dots, 1),$$

for a proof see [9, Lemma 2.2.19].

Since $Dw = 0$,

$$(5.18) \quad D \log h_n^n = -\lambda D\chi,$$

hence

$$(5.19) \quad F^{ij}(\log h_n^n)_i(\log h_n^n)_j \leq \lambda^2 F^{ij}\chi_i\chi_j.$$

Hence, we conclude that κ_n is a priori bounded in this case.

Case 2. Suppose that

$$(5.20) \quad \kappa_1 \geq -\epsilon_1 \kappa_n,$$

then, the last term in inequality (5.14) is estimated from above by

$$(5.21) \quad \begin{aligned} & \frac{2}{1 + \epsilon_1} \sum_{i=1}^n (F_n - F_i)(h_{ni};^n)^2 (h_n^n)^{-2} \\ & \leq \frac{2}{1 + 2\epsilon_1} \sum_{i=1}^n (F_n - F_i)(h_{nn};^i)^2 (h_n^n)^{-2} \\ & \quad + c(\epsilon_1) \sum_{i=1}^n (F_i - F_n)\kappa_n^{-2}, \end{aligned}$$

where we used the Codazzi equation. The last sum can be easily balanced.

The terms in (5.14) containing the derivative of h_n^n can therefore be estimated from above by

$$(5.22) \quad \begin{aligned} & -\frac{1 - 2\epsilon_1}{1 + 2\epsilon_1} \sum_{i=1}^n F_i (h_{nn};^i)^2 (h_n^n)^{-2} \\ & + \frac{2}{1 + 2\epsilon_1} F_n \sum_{i=1}^n (h_{nn};^i)^2 (h_n^n)^{-2} \\ & \leq 2F_n \sum_{i=1}^n (h_{nn};^i)^2 (h_n^n)^{-2} \\ & = 2\lambda^2 F_n \|D\chi\|^2. \end{aligned}$$

Hence, we infer

$$(5.23) \quad \begin{aligned} 0 \leq & -\frac{1}{2}F_n\kappa_n^2 + \lambda^2 cF_n + c\kappa_n + cF^{ij}g_{ij} \\ & + \lambda c - \lambda c_0 F^{ij}g_{ij}. \end{aligned}$$

From (2.21), (2.22) and Lemma 5.1 we deduce

$$(5.24) \quad F^{ij}g_{ij} \geq c\kappa_n,$$

thus, taking (5.17) into account, we obtain an a priori estimate

$$(5.25) \quad \kappa_n \leq \text{const},$$

if λ is chosen large enough. Notice that ϵ_1 is only subject to the requirement $0 < \epsilon_1 < \frac{1}{2}$. \square

Remark 5.4. In view of (0.5) and (3.8), we conclude that the principal curvatures of the flow hypersurfaces stay in a compact subset of Γ .

6. EXISTENCE OF A SOLUTION

We shall show that the solution of the evolution problem (3.1) exists for all time, and that it converges to a stationary solution.

Proposition 6.1. *The solutions $M(t) = \text{graph } u(t)$ of the evolution problem (3.1) with $F = \sigma_2$ and $M(0) = M_2$ exist for all time and converge to a stationary solution provided $f \in C^{4,\alpha}$ satisfies the conditions (0.5), (4.22) and (4.23).*

Proof. Let us look at the scalar version of the flow as in (4.6)

$$(6.1) \quad \frac{\partial u}{\partial t} = -e^{-\psi}v(F - f).$$

This is a scalar parabolic differential equation defined on the cylinder

$$(6.2) \quad Q_{T^*} = [0, T^*) \times \mathcal{S}_0$$

with initial value $u(0) = u_2 \in C^{4,\alpha}(\mathcal{S}_0)$.

In view of the a priori estimates, which we have established in the preceding sections, we know that

$$(6.3) \quad |u|_{2,0,\mathcal{S}_0} \leq c$$

and

$$(6.4) \quad F \text{ is uniformly elliptic in } u$$

independently of t , in view of Remark 5.4. Thus, we can apply the known regularity results, see e.g. [14, Chapter 5.5], where even more general operators are considered, to conclude that uniform $C^{2,\alpha}$ -estimates are valid. Therefore, the maximal time interval is unbounded, i.e. $T^* = \infty$.

Now, integrating (6.1) with respect to t , and observing that the right-hand side is non-positive, yields

$$(6.5) \quad u(0, x) - u(t, x) = \int_0^t e^{-\psi}v(F - f) \geq c \int_0^t (F - f),$$

i.e.,

$$(6.6) \quad \int_0^\infty |F - f| < \infty \quad \forall x \in \mathcal{S}_0.$$

Hence, for any $x \in \mathcal{S}_0$ there is a sequence $t_k \rightarrow \infty$ such that $(F - f) \rightarrow 0$.

On the other hand, $u(\cdot, x)$ is monotone decreasing and therefore

$$(6.7) \quad \lim_{t \rightarrow \infty} u(t, x) = \tilde{u}(x)$$

exists and is of class $C^{2,\alpha}(\mathcal{S}_0)$. We conclude that \tilde{u} is a stationary solution, and that

$$(6.8) \quad \lim_{t \rightarrow \infty} (F - f) = 0.$$

Now, we can deduce that uniform $C^{6,\alpha}$ -estimates are valid, cf [11, Theorem 6.5]. Hence, we conclude that the functions $u(t, \cdot)$ converge in $C^6(\mathcal{S}_0)$ to $\tilde{u} \in C^{6,\alpha}(\mathcal{S}_0)$. \square

We want to solve the equation

$$(6.9) \quad \sigma_{2|M} = f^{\frac{1}{2}}(x, \nu),$$

where f satisfies the conditions of (0.5), (0.6) and (0.7). Thus we would like to apply the preceding existence result. But, unfortunately, the derivatives f_β resp. $f_{\nu\beta}$ grow quadratically resp. linear in $\|\nu\|$ contrary to the assumptions in Proposition 6.1. Therefore, we define a smooth cut-off function $\theta \in C^\infty(\mathbb{R}_+)$, $0 < \theta \leq 2k$, where $k \geq k_0 > 1$ is to be determined later, by

$$(6.10) \quad \theta(t) = \begin{cases} t, & 0 \leq t \leq k, \\ 2k, & 2k \leq t, \end{cases}$$

such that

$$(6.11) \quad 0 \leq \dot{\theta} \leq 4$$

and consider the problem

$$(6.12) \quad \sigma_{2|M} = f^{\frac{1}{2}}(x, \tilde{\nu}),$$

where for a spacelike hypersurface $M = \text{graph } u$ with past directed normal vector ν we set

$$(6.13) \quad \tilde{\nu} = \theta(\tilde{\nu})\tilde{\nu}^{-1}\nu.$$

Then

$$(6.14) \quad \|\tilde{\nu}\| \leq ck,$$

so that the assumptions in Proposition 6.1 are certainly satisfied. The constant k_0 should be so large that $\tilde{\nu} = \nu$ in case of the barriers $M_i, i = 1, 2$. Proposition 6.1 is therefore applicable leading to a solution $M_k = \text{graph } u_k$ of (6.12).

From [7, Lemma 8.1] we then deduce that there exists a constant m such that

$$(6.15) \quad \tilde{\nu} = (1 - |Du_k|^2)^{-\frac{1}{2}} \leq m \quad \forall k.$$

Hence, $M_k = \text{graph } u_k$ is a solution of (0.4), if we choose $k \geq \max(2m, k_0)$.

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